Transformation between the Young - Yamanouchi basis and its dual

1996 J. Phys. A: Math. Gen. 29 5935

(http://iopscience.iop.org/0305-4470/29/18/022)

J. Phys. A: Math. Gen. **29** (1996) 5935–5944. Printed in the UK

Transformation between the Young–Yamanouchi basis and its dual

A M Hamel†, L F McAven‡, H J Ross‡ and P H Butler‡

† Department of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand

‡ Department of Physics and Astronomy, University of Canterbury, Christchurch, New Zealand

Received 4 April 1996

Abstract. Motivated by the aim of finding generalized transformation coefficients for the symmetric group, we calculate the matrix which transforms the basis functions of the Young–Yamanouchi basis into the basis functions of its dual. Our approach is to derive the representation matrices for both bases and then determine the transformation matrix. The dual basis is associated with the subgroup chain $S_1 \times S_{n-1} \supset S_1 \times S_1 \times S_{n-2} \supset \cdots$, whereas the usual YY basis is associated with the subgroup chain $S_{n-1} \times S_1 \supset S_{n-2} \times S_1 \times S_1 \supset \cdots$. A combinatorial technique, *jeu de taquin*, is used to define the $\overline{\mathrm{YY}}$ basis, via the Young–Yamanouchi symbols and Young tableaux with which the basis functions can be indexed.

1. Introduction

Representations of the symmetric group, S_n, the associated matrices, characters, and basis functions, play an important role in the study of the many-electron problem in physics and quantum chemistry. A common choice among the wide range of bases is the Young–Yamanouchi, or YY basis (see [1–4]), associated with the subgroup chain $S_{n-1} \times S_1 \supset S_{n-2} \times S_1 \times S_1 \supset \cdots \supset S_1 \times S_1 \times \cdots \times S_1$. The aim of finding generalized transformation coefficients for the symmetric group motivates us to calculate a special case. We calculate the matrix that transforms between the basis functions of the YY basis and what we shall call its dual, the $\overline{\mathrm{YY}}$ basis. This basis is associated with the subgroup chain $S_1 \times S_{n-1} \supset S_1 \times S_1 \times S_{n-2} \supset \cdots \supset S_1 \times S_1 \times \cdots \times S_1$.

The YY basis is defined by a chain of maximal subgroups. Restrictions of generic basis functions of irreps of S_n will give a non-reduced basis for subgroups. The YY basis functions corresponding to the irreducible representations (irreps) of S_n are also basis functions of irreps of the subgroups $S_{n-1} \times S_1, S_{n-2} \times S_1 \times S_1, \ldots$. The irreps of such a direct product group can be expressed as the direct product of irreps of the factor groups. The only irrep of S_1 corresponds to the one-dimensional unit matrix which is 1. Therefore the irreps of the subgroups in the YY basis can be simply labelled using the first factor of the subgroup. Each function can thus be identified with the irreps to which it will belong in $S_n, S_{n-1}, S_{n-2}, \ldots$, and this identification corresponds to the unique Young tableau with the property that the successive removal of the boxes labelled $n, n-1, \ldots$ yields Young tableaux that correspond to the irreps of S_{n-1}, etc.

A more general set of basis functions would correspond to the basis $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_l}$, $n_1 + n_2 + \cdots + n_l = n$. In this subgroup basis the matrices of elements in the subgroup are direct sums of tensor products of matrix irreps of the factor groups (up to a permutation

0305-4470/96/185935+10$19.50

of the basis). The YY basis is the specific case of this where $n_i = 1$, $\forall i > 1$. We want to look at transformations between the YY basis and the more general basis where the n_i, $i > 1$ need not be equal to 1.

This problem is equivalent to looking for the matrix that transforms between $S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k}$ and $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_l}$ where $m_1 + m_2 + \cdots + m_k = n_1 + n_2 + \cdots n_l = n$. This general problem has been considered by [4–6] among others. However, as Horie's method is recursive, and Suryanarayana and Kondala Rao have a closed formula only for representations of the form $\lambda = 2^a 1^{n-2a}$, there is still a need for other methods. The techniques we present here provide a straightforward and easily explained approach to a specific sort of basis transformation, and we can avoid the Young operator techniques employed in other approaches.

In this paper we determine the transformation matrix for a specific case of the general transformation mentioned above. That is, for the transformation between the YY basis and its dual. Our approach is to derive the representation matrices for both bases and then determine the transformation matrix. Since any permutation can be expressed as the product of adjacent transpositions, $(k, k + 1)$, we can confine our discussion to the representation matrices for these. The representation matrices corresponding to the YY basis functions are well known (see, for example, [3, 4, 7, 8]); those corresponding to the $\overline{\text{YY}}$ basis can be constructed using the same approach after defining the basis using the combinatorial technique of *jeu de taquin* due to [9, 10].

2. Indexing the basis functions using *jeu de taquin* and the Young–Yamanouchi symbols

The well known Young–Yamanouchi, or YY, symbols (described below) are generated from tableaux by removing one box at a time from the Young tableaux, starting with the box labelled n. To derive the $\overline{\text{YY}}$ symbols for the $\overline{\text{YY}}$ basis functions we remove the boxes from the Young tableau one at a time starting with 1 instead of n and filling the holes with *jeu de taquin* at each stage. *Jeu de taquin* is a combinatorial technique that provides a means of indexing the $\overline{\text{YY}}$ basis functions with a sequence of integers. This technique is due to [9, 10] and is equivalent to the Robinson–Schensted algorithm [11]. *Jeu de taquin* is a procedure for removing boxes from any part of a Young tableau (not just the perimeter) and filling the gap created so that the resulting object is a proper Young tableau. We describe it here first, later using it to define the representation matrices in the $\overline{\text{YY}}$ basis.

> Remove a box from the Young tableau. Examine the content of the box to the right of the removed box and that of the box below the removed box. Slide the box containing the smaller of these two numbers into the vacant position. Now repeat this procedure to fill the hole created by the slide. Repeat the entire process until no holes remain (i.e. the hole has worked itself to the perimeter of the tableau).

The tableaux of the YY basis are uniquely indexed by a sequence of integers, a so-called *Young–Yamanouchi symbol*, defined in the following manner.

> Locate the box containing n in the Young tableau. Remove that box and write down the index of the row that contained that box. Repeat the procedure for $n - 1, n - 2, \ldots, 2, 1$. The list of integers is the YY symbol.

The basis functions of the $\overline{\text{YY}}$ basis are also indexed by Young tableaux, and can similarly be indexed by a sequence of integers defined in the following manner.

> Locate the box containing 1 in the Young tableau. Remove it and fill the hole it leaves using *jeu de taquin*. Write down the index of the row from which a box is ultimately removed after the *jeu de taquin*. Repeat this procedure for $2, 3, \ldots, n$. The list of integers is the $\overline{\mathrm{YY}}$ symbol.

The set of $\overline{\mathrm{YY}}$ symbols is, in fact, identical to the set of YY symbols. However, a given Young tableau will, in general, have different YY and $\overline{\mathrm{YY}}$ symbols. We say a YY symbol and a $\overline{\mathrm{YY}}$ symbol correspond if they are both generated by the same Young tableau.

We define a companion tableaux, $\tilde{T}$, of T to be a tableaux such that the $\overline{\mathrm{YY}}$ symbol of $\tilde{T}$ is equal to the YY symbol of T. The companion relation is a symmetric relation, i.e. the YY symbol of $\tilde{T}$ is equal to the $\overline{\mathrm{YY}}$ symbol of T. We note that there is an object called the dual tableau (see [12] pp 58–9 for details) which is defined such that the YY symbols for it and for the original tableau are the same; however, the dual has the filling rules reversed (i.e. each entry in the dual tableau has to be greater than the entry to the left of or below it) and hence it would be necessary to use *jeu de tacquin* to obtain the YY symbol. The dual tableau is, however, equivalent to our definition of companion tableau, and the correspondence can be simply seen by reversing the order of the labels and the order in which we remove the labels. The use of the companion rather than the dual has the obvious advantage that the companion is within the original set of standard tableaux.

3. Representation matrices

The representation matrices indexed by the YY basis functions for adjacent transpositions are well known [2, 3]. Descriptions of the entries are in terms of a tableau parameter called *axial distance* and defined as follows. Let i be the box of the tableau in row r_i and column c_i, and let j be the box of the tableau in row r_j and columns c_j. Then the axial distance from i to j is $(c_j - r_j) - (c_i - r_i) \equiv \tau_{ij}$. We also define the reciprocal of this, $\rho := 1/\tau$.

Since we will be indexing matrices by Young tableaux, we first impose a total order on the tableaux. Let λ be a partition. Take the set of all Young tableaux of shape λ, and order them using last-letter order, i.e. tableaux in which the largest letter occurs in a lower row are later in the ordering. For example, for $\lambda = 3, 2$, see figure 3. The well-defined order of the tableaux imposes an ordering upon the associated YY symbols and their corresponding $\overline{\mathrm{YY}}$ symbols. When we refer to the ordering of symbols it is this order to which we refer. The ordering of a set of tableaux, YY, and $\overline{\mathrm{YY}}$ symbols are given in figure 2.

The YY representation matrix $M^{\lambda}_{(k-1,k)}$ for the transposition $(k-1, k)$ for the representation λ is defined as follows:

(1) $M^{\lambda}_{(k-1,k)}$ has $+1$ in position (r, r) if in the rth YY symbol, the $(n-k+1)$-th and $(n-k)$th elements are identical (i.e. the rth tableau has $k-1$ and k in the same row).

(2) $M^{\lambda}_{(k-1,k)}$ has a -1 in position (r, r) if in the rth YY symbol the $(n-k+1)$-th element, α, is one more than the $(n-k)$-th element, β, and there does *not* exist another YY symbol that is identical to this YY symbol except that its $(n-k+1)$-th element is β and its $(n-k)$-th element is α (i.e. the rth tableau has $k-1$ and k in the same column).

(3) $M^{\lambda}_{(k-1,k)}$ has $-\rho$ in position (r, r), $\sqrt{1-\rho^2}$ in positions (r, s) and (s, r) and ρ in position (s, s), if $r < s$ and the rth and sth YY symbols are identical except that the $(n-k+1)$-th element of the rth YY symbol is the $(n-k)$-th element of the sth YY symbol and vice versa (i.e. the sth tableau is obtained from the rth tableau by interchanging $k-1$ and k). As noted above, ρ is the reciprocal of the axial distance from the box containing $k-1$ to the box containing k in the rth tableau.

(4) 0 in all other positions.

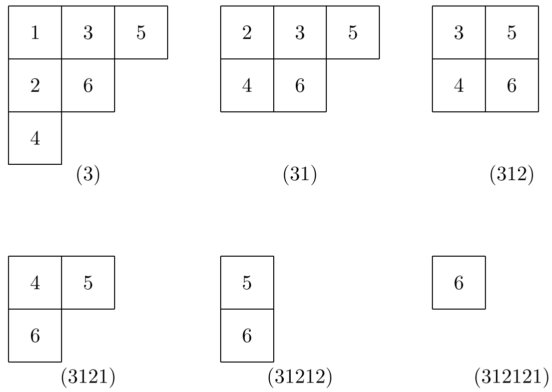

Figure 1. Example of *jeu de taquin* on $\lambda = 3, 2, 1$. The $\overline{\mathrm{YY}}$ symbol is built up progressively underneath the tableau at each stage.

(See [2, pp 41–9]; also [13, ch VI, pp 217–18]). An example is given in the appendix.

The $\overline{\mathrm{YY}}$ representation matrix $\overline{M}^{\lambda}_{(k-1,k)}$ for the transposition $(k-1, k)$ for the representation λ is defined as follows:

(1) $\overline{M}^{\lambda}_{(k-1,k)}$ has $+1$ in position (r, r) if in the rth $\overline{\mathrm{YY}}$ symbol the $(k-1)$-th and kth elements are identical.

(2) $\overline{M}^{\lambda}_{(k-1,k)}$ has a -1 in position (r, r) if in the rth $\overline{\mathrm{YY}}$ symbol the $(k-1)$-th element, α, is one more than the kth element, β, and there does not exist another $\overline{\mathrm{YY}}$ symbol that is identical to this $\overline{\mathrm{YY}}$ symbol except that its $(k-1)$-th element is β and its kth element is α.

(3) $\overline{M}^{\lambda}_{(k-1,k)}$ has $-\rho$ in position (r, r), $\sqrt{1-\rho^2}$ in positions (r, s) and (s, r) and ρ in position (s, s), if $r < s$ the rth and sth $\overline{\mathrm{YY}}$ symbols are identical except that the $(k-1)$-th element of the rth $\overline{\mathrm{YY}}$ symbol is the kth element of the sth $\overline{\mathrm{YY}}$ symbol and vice versa, where ρ is the reciprocal of the axial distance from $n-k+2$ to $n-k+1$ in the rth tableau.

(4) 0 in all other positions.

For an example, see the appendix.

The $\overline{M}^{\lambda}_{(k-1,k)}$ matrix is not as well known in the literature, although it has appeared in [4, p 52] for a special case, and is easily derived by induction in a manner similar to [2, pp 39–43] or [3, pp 215–23]. In doing so we can choose the representation matrices of the $\overline{\mathrm{YY}}$ basis to be related to the representation matrices of the YY basis by a permutation of the basis functions,

$$P\overline{M}^{\lambda}_{(k-1,k)}P^{-1} = M^{\lambda}_{(n-k+2,n-k+1)}. \tag{1}$$

At the end of the previous section we stated that the lists of $\overline{\mathrm{YY}}$ and YY symbols, ordered by the last-letter order of the tableaux, contain the same elements, possibly in a different order. If we call those lists x and y, for the YY and $\overline{\mathrm{YY}}$ symbols, respectively, then

$$\begin{aligned} P_{i,j} &= \delta(x_i, y_j) \\ &= \delta(x_j, y_i) \end{aligned}$$

where δ is the Kronecker δ function: $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ if $a \neq b$. Thus we say that the ordered list of $\overline{\mathrm{YY}}$ symbols is carried to the ordered list of YY symbols by the permutation P, and that P takes each tableau to its companion tableau.

Tableau	YY symbol	$\overline{YY}$ symbol	Tableau	YY symbol	$\overline{YY}$ symbol
1 2 3 4 5 6	322111	123121	1 2 5 3 6 4	213211	132121
1 2 4 3 5 6	321211	121321	1 3 5 2 6 4	213121	312121
1 3 4 2 5 6	321121	211321	1 4 5 2 6 3	211321	321121
1 2 5 3 4 6	312211	231121	1 2 6 3 4 5	132211	231211
1 3 5 2 4 6	312121	213121	1 3 6 2 4 5	132121	213211
1 2 3 4 6 5	232111	123211	1 2 6 3 5 4	123211	232111
1 2 4 3 6 5	231211	132211	1 3 6 2 5 4	123121	322111
1 3 4 2 6 5	231121	312211	1 4 6 2 5 3	121321	321211

Figure 2. Tableaux, YY symbols, and $\overline{\mathrm{YY}}$ symbols for $\lambda = 3, 2, 1$.

4. Calculation of the transformation matrix

The transformation matrix is the matrix T such that

$$\overline{M}^{\lambda}_{(k-1,k)} = T^{-1} M^{\lambda}_{(k-1,k)} T \qquad \forall\ k. \tag{2}$$

It can be calculated simply in two stages and expressed as the product of two matrices, P and Q. The first of these, the P matrix, is simply the permutation matrix that sends the set of $\overline{\mathrm{YY}}$ symbols to the set of YY symbols. The application of P and P^{-1} to the right and left sides of $\overline{M}^{\lambda}_{(k-1,k)}$ brings it to the same form as $M^{\lambda}_{(n-k+2,n-k+1)}$.

$$\begin{aligned} P^{-1} Q^{-1} M^{\lambda}_{(k-1,k)} Q P &= \overline{M}^{\lambda}_{(k-1,k)} \\ Q^{-1} M^{\lambda}_{(k-1,k)} Q &= P \overline{M}^{\lambda}_{(k-1,k)} P^{-1} \\ &= M^{\lambda}_{(n-k+2,n-k+1)} \end{aligned}$$

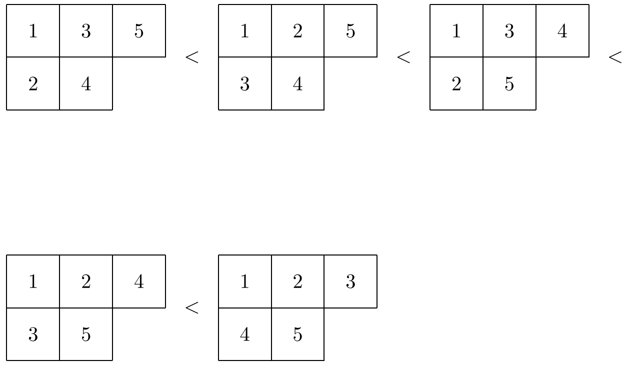

Figure 3. Example of tableaux of shape $\lambda = 3, 2$ ordered by last-letter order.

as in equation (2), so that

$$Q^{-1} M^{\lambda}_{(k-1,k)} Q = M^{\lambda}_{(n-k+2,n-k+1)}. \tag{3}$$

The problem then reduces to finding a transformation matrix Q between representation matrices in the YY basis.

It is easily seen that the Q matrix is the representation matrix that sends $n, n-1, \ldots, 2, 1$ to $1, 2, \ldots, n-1, n$. Q can be calculated directly from the representation matrices in the YY basis for the adjacent transpositions, since it is well known that any permutation σ can be expressed as a minimal length product of adjacent transpositions [2, p 6]. This minimal length is the number of inversions, i.e. the number of distinct pairs (i, j) with $i < j$ such that $\sigma(i) > \sigma(j)$. In particular, define $d_i = \mathrm{card}\{j | j > k$ where $\sigma(k) = i$ and $\sigma(j) < i\}$. Then the permutation σ can be written as

$$\sigma = \ldots(\tau_{i-1}\tau_{i-2}\ldots\tau_{i-d_i})\ldots(\tau_{n-2}\tau_{n-3}\ldots\tau_{n-1-d_{n-1}})(\tau_{n-1}\tau_{n-2}\ldots\tau_{n-d_n})$$

where $\tau_i = (i, i+1)$ and the ith contribution is included only if $d_i \geqslant 1$.

In the case of Q, the length of the product will be $\binom{n}{2}$ and $d_i = i - 1$, $1 \leqslant i \leqslant n$. Specifically, then,

$$Q = \prod_{i=2}^{n} \prod_{j=i-1}^{1} (j, j+1).$$

For our example of $\lambda = 3, 2, 1$,

$$\begin{aligned} Q &= (16)(25)(34) \\ &= (12)(23)(12)(34)(23)(12)(45)(34)(23)(12)(56)(45)(34)(23)(12) \end{aligned}$$

and the matrices P and Q are given in figures 4 and 5, respectively.

5. Conclusion

We have presented a simple, straightforward method for calculating the transformation matrix between the YY basis and its dual, the $\overline{\mathrm{YY}}$ basis. The matrix itself is easy to describe, is intuitively pleasing, and is presented without the use of Young operator techniques. By using the well known combinatorial technique of *jeu de taquin*, the method presented here further cements the link between combinatorics and mathematical physics. We anticipate that further links are possible, and that using variations on the methods presented here, we

$$
P=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
$$

Figure 4. The P matrix for $\lambda = 3, 2, 1$.

$$
\begin{bmatrix}
0 & \frac{1}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & 0 & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & \frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & 0 & \frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{3\sqrt{10}}{16} & 0\\
\frac{1}{8\sqrt{2}} & \frac{1}{16} & \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & -\frac{3}{16} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & \frac{3}{8} & -\frac{\sqrt{5}}{16} & 0 & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{5}}{8} & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{16} & 0 & \frac{3\sqrt{10}}{16}\\
\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{8} & -\frac{3}{16} & \frac{5}{16} & 0 & -\frac{1}{8\sqrt{2}} & -\frac{1}{8} & \frac{\sqrt{3}}{16} & 0 & \frac{3\sqrt{5}}{16} & \frac{3\sqrt{10}}{16} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}}\\
\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{8} & \frac{5}{16} & -\frac{3}{16} & 0 & \frac{3}{8\sqrt{2}} & \frac{3}{8} & \frac{5\sqrt{3}}{16} & 0 & \frac{\sqrt{5}}{16} & -\frac{\sqrt{10}}{16} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}}\\
\frac{3}{8\sqrt{2}} & -\frac{3}{16} & 0 & 0 & \frac{5}{16} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & 0 & \frac{3\sqrt{5}}{16} & \frac{\sqrt{15}}{8} & -\frac{\sqrt{15}}{8\sqrt{2}} & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{5}}{8} & \frac{\sqrt{10}}{16}\\
0 & \frac{\sqrt{3}}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & 0 & \frac{3}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{10}}{16} & 0 & \frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{10}}{16} & \frac{3\sqrt{10}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & 0\\
\frac{\sqrt{3}}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & -\frac{1}{8} & \frac{3}{8} & \frac{\sqrt{3}}{16} & \frac{3}{8\sqrt{2}} & \frac{3}{16} & -\frac{\sqrt{3}}{8} & -\frac{\sqrt{15}}{16} & 0 & -\frac{\sqrt{10}}{16} & \frac{\sqrt{15}}{8} & \frac{\sqrt{5}}{16} & -\frac{3\sqrt{5}}{16} & 0 & -\frac{\sqrt{15}}{8\sqrt{2}}\\
\frac{3}{8\sqrt{2}} & \frac{3}{8} & \frac{\sqrt{3}}{16} & \frac{5\sqrt{3}}{16} & 0 & -\frac{\sqrt{3}}{8\sqrt{2}} & -\frac{\sqrt{3}}{8} & -\frac{1}{16} & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{3\sqrt{5}}{16} & 0 & 0 & \frac{\sqrt{5}}{16} & \frac{\sqrt{10}}{16}\\
\frac{\sqrt{10}}{16} & -\frac{\sqrt{5}}{16} & 0 & 0 & \frac{3\sqrt{5}}{16} & \frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{15}}{16} & 0 & \frac{1}{16} & -\frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8\sqrt{2}} & 0 & \frac{5\sqrt{3}}{16} & \frac{\sqrt{3}}{16} & -\frac{3}{8} & \frac{3}{8\sqrt{2}}\\
\frac{\sqrt{15}}{8\sqrt{2}} & 0 & \frac{3\sqrt{5}}{16} & -\frac{\sqrt{5}}{16} & \frac{\sqrt{15}}{8} & -\frac{\sqrt{10}}{16} & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{3}}{8} & -\frac{3}{16} & \frac{3}{8\sqrt{2}} & \frac{\sqrt{3}}{16} & -\frac{3}{8} & \frac{1}{8} & \frac{\sqrt{3}}{16} & -\frac{\sqrt{3}}{8\sqrt{2}}\\
0 & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{3\sqrt{10}}{16} & -\frac{\sqrt{10}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & 0 & -\frac{\sqrt{10}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & 0 & \frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{8\sqrt{2}} & -\frac{1}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & 0\\
\frac{\sqrt{10}}{16} & \frac{\sqrt{5}}{8} & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{16} & 0 & \frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{15}}{8} & -\frac{3\sqrt{5}}{16} & 0 & \frac{\sqrt{3}}{16} & \frac{\sqrt{3}}{8\sqrt{2}} & -\frac{5}{16} & 0 & 0 & \frac{3}{16} & \frac{3}{8\sqrt{2}}\\
\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{10}}{16} & \frac{\sqrt{5}}{16} & 0 & \frac{5\sqrt{3}}{16} & -\frac{3}{8} & \frac{3}{8\sqrt{2}} & 0 & \frac{3}{16} & -\frac{5}{16} & \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{8\sqrt{2}}\\
\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{15}}{16} & 0 & 0 & -\frac{\sqrt{15}}{16} & \frac{3\sqrt{10}}{16} & -\frac{3\sqrt{5}}{16} & 0 & \frac{\sqrt{3}}{16} & \frac{1}{8} & -\frac{1}{8\sqrt{2}} & 0 & -\frac{5}{16} & \frac{3}{16} & \frac{\sqrt{3}}{8} & -\frac{\sqrt{3}}{8\sqrt{2}}\\
\frac{3\sqrt{10}}{16} & 0 & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{15}}{16} & -\frac{\sqrt{5}}{8} & -\frac{\sqrt{15}}{8\sqrt{2}} & 0 & \frac{\sqrt{5}}{16} & -\frac{3}{8} & \frac{\sqrt{3}}{16} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{3}{16} & \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{8} & -\frac{1}{16} & \frac{1}{8\sqrt{2}}\\
0 & \frac{3\sqrt{10}}{16} & -\frac{\sqrt{15}}{8\sqrt{2}} & -\frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{10}}{16} & 0 & -\frac{\sqrt{15}}{8\sqrt{2}} & \frac{\sqrt{10}}{16} & \frac{3}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & 0 & \frac{3}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & -\frac{\sqrt{3}}{8\sqrt{2}} & \frac{1}{8\sqrt{2}} & 0
\end{bmatrix}
$$

Figure 5. The Q matrix for $\lambda = 3, 2, 1$ calculated using Matlab.

would be able to calculate transformation matrices between more general bases of symmetric groups, e.g. between two bases of S_n of the form $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_l}$, $n_1 + n + 2 + \cdots + n_l = n$. This will be the subject of future work.

Acknowledgments

The first author wishes to acknowledge the support of this research initially by the Natural Sciences and Engineering Research Council (NSERC) of Canada, and subsequently by the

Foundation for Research, Science, and Technology (FRST) of New Zealand.

6. Appendix

Representation matrices for $\lambda = 3, 2, 1$ in the YY basis. The basis functions are ordered according to the last-letter order of the tableaux, as given in figure 2.

$$(12) = \begin{vmatrix}
1 \\
& 1 \\
&& -1 \\
&&& 1 \\
&&&& -1 \\
&&&&& 1 \\
&&&&&& 1 \\
&&&&&&& -1 \\
&&&&&&&& 1 \\
&&&&&&&&& -1 \\
&&&&&&&&&& -1 \\
&&&&&&&&&&& 1 \\
&&&&&&&&&&&& -1 \\
&&&&&&&&&&&&& 1 \\
&&&&&&&&&&&&&& -1 \\
&&&&&&&&&&&&&&& -1
\end{vmatrix}$$

$$(23) = \begin{vmatrix}
1 \\
& -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
& \frac{\sqrt{3}}{2} & \frac{1}{2} \\
&&& -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
&&& \frac{\sqrt{3}}{2} & \frac{1}{2} \\
&&&&& 1 \\
&&&&&& -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
&&&&&& \frac{\sqrt{3}}{2} & \frac{1}{2} \\
&&&&&&&& -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
&&&&&&&& \frac{\sqrt{3}}{2} & \frac{1}{2} \\
&&&&&&&&&& -1 \\
&&&&&&&&&&& -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
&&&&&&&&&&& \frac{\sqrt{3}}{2} & \frac{1}{2} \\
&&&&&&&&&&&&& -\frac{1}{2} & \frac{\sqrt{3}}{2} \\
&&&&&&&&&&&&& \frac{\sqrt{3}}{2} & \frac{1}{2} \\
&&&&&&&&&&&&&&& -1
\end{vmatrix}$$

$$(34) = \begin{vmatrix}
-\frac{1}{3} & \frac{\sqrt{8}}{3} \\
\frac{\sqrt{8}}{3} & \frac{1}{3} \\
&& 1 \\
&&& 1 \\
&&&& -1 \\
&&&&& -\frac{1}{3} & \frac{\sqrt{8}}{3} \\
&&&&& \frac{\sqrt{8}}{3} & \frac{1}{3} \\
&&&&&&& 1 \\
&&&&&&&& -1 \\
&&&&&&&&& -\frac{1}{3} & \frac{\sqrt{8}}{3} \\
&&&&&&&&& \frac{\sqrt{8}}{3} & \frac{1}{3} \\
&&&&&&&&&&& 1 \\
&&&&&&&&&&&& -1 \\
&&&&&&&&&&&&& -1 \\
&&&&&&&&&&&&&& -\frac{1}{3} & \frac{\sqrt{8}}{3} \\
&&&&&&&&&&&&&& \frac{\sqrt{8}}{3} & \frac{1}{3}
\end{vmatrix}$$

$$(45) = \begin{vmatrix}
1 \\
& -\frac{1}{2} && \frac{\sqrt{3}}{2} \\
&& -\frac{1}{2} && \frac{\sqrt{3}}{2} \\
& \frac{\sqrt{3}}{2} && \frac{1}{2} \\
&& \frac{\sqrt{3}}{2} && \frac{1}{2} \\
&&&&& -1 \\
&&&&&& -\frac{1}{4} && \frac{\sqrt{15}}{4} \\
&&&&&&& -\frac{1}{4} && \frac{\sqrt{15}}{4} \\
&&&&&& \frac{\sqrt{15}}{4} && \frac{1}{4} \\
&&&&&&& \frac{\sqrt{15}}{4} && \frac{1}{4} \\
&&&&&&&&&& 1 \\
&&&&&&&&&&& -\frac{1}{2} && \frac{\sqrt{3}}{2} \\
&&&&&&&&&&&& -\frac{1}{2} && \frac{\sqrt{3}}{2} \\
&&&&&&&&&&& \frac{\sqrt{3}}{2} && \frac{1}{2} \\
&&&&&&&&&&&& \frac{\sqrt{3}}{2} && \frac{1}{2} \\
&&&&&&&&&&&&&&& -1
\end{vmatrix}$$

$$
(56) = \begin{vmatrix}
-\frac{1}{2} & & & & & \frac{\sqrt{3}}{2} & & & & & & & & & & \\
& -\frac{1}{2} & & & & & \frac{\sqrt{3}}{2} & & & & & & & & & \\
& & -\frac{1}{2} & & & & & \frac{\sqrt{3}}{2} & & & & & & & & \\
& & & -\frac{1}{4} & & & & & & & & \frac{\sqrt{15}}{4} & & & & \\
& & & & -\frac{1}{4} & & & & & & & & \frac{\sqrt{15}}{4} & & & \\
\frac{\sqrt{3}}{2} & & & & & \frac{1}{2} & & & & & & & & & & \\
& \frac{\sqrt{3}}{2} & & & & & \frac{1}{2} & & & & & & & & & \\
& & \frac{\sqrt{3}}{2} & & & & & \frac{1}{2} & & & & & & & & \\
& & & & & & & & -\frac{1}{2} & & & & & \frac{\sqrt{3}}{2} & & \\
& & & & & & & & & -\frac{1}{2} & & & & & \frac{\sqrt{3}}{2} & \\
& & & & & & & & & & -\frac{1}{2} & & & & & \frac{\sqrt{3}}{2} \\
& & & \frac{\sqrt{15}}{4} & & & & & & & & \frac{1}{4} & & & & \\
& & & & \frac{\sqrt{15}}{4} & & & & & & & & \frac{1}{4} & & & \\
& & & & & & & & \frac{\sqrt{3}}{2} & & & & & \frac{1}{2} & & \\
& & & & & & & & & \frac{\sqrt{3}}{2} & & & & & \frac{1}{2} & \\
& & & & & & & & & & \frac{\sqrt{3}}{2} & & & & & \frac{1}{2}
\end{vmatrix}
$$

Representation matrices for $\lambda = 3, 2, 1$ in the $\overline{\mathrm{YY}}$ basis. The basis functions are ordered according to the last-letter order of the tableaux, as given in figure 2.

$$
(12) = \begin{vmatrix}
\frac{1}{2} & & & & \frac{\sqrt{3}}{2} & & & & & & & & & & & \\
& \frac{1}{2} & \frac{\sqrt{3}}{2} & & & & & & & & & & & & & \\
& \frac{\sqrt{3}}{2} & -\frac{1}{2} & & & & & & & & & & & & & \\
& & & \frac{1}{2} & & & & & & & \frac{\sqrt{3}}{2} & & & & & \\
\frac{\sqrt{3}}{2} & & & & -\frac{1}{2} & & & & & & & & & & & \\
& & & & & \frac{1}{2} & & & & & & & \frac{\sqrt{3}}{2} & & & \\
& & & & & & \frac{1}{4} & \frac{\sqrt{15}}{4} & & & & & & & & \\
& & & & & & \frac{\sqrt{15}}{4} & -\frac{1}{4} & & & & & & & & \\
& & & & & & & & \frac{1}{4} & \frac{\sqrt{15}}{4} & & & & & & \\
& & & & & & & & \frac{\sqrt{15}}{4} & -\frac{1}{4} & & & & & & \\
& & & \frac{\sqrt{3}}{2} & & & & & & & -\frac{1}{2} & & & & & \\
& & & & & & & & & & & \frac{1}{2} & & & & \frac{\sqrt{3}}{2} \\
& & & & & \frac{\sqrt{3}}{2} & & & & & & & -\frac{1}{2} & & & \\
& & & & & & & & & & & & & \frac{1}{2} & \frac{\sqrt{3}}{2} & \\
& & & & & & & & & & & & & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \\
& & & & & & & & & & & \frac{\sqrt{3}}{2} & & & & -\frac{1}{2}
\end{vmatrix}
$$

$$
(23) = \begin{vmatrix}
\frac{1}{2} & & & & & & & & \frac{\sqrt{3}}{2} & & & & & & & \\
& -1 & & & & & & & & & & & & & & \\
& & 1 & & & & & & & & & & & & & \\
& & & -\frac{1}{4} & \frac{\sqrt{15}}{4} & & & & & & & & & & & \\
& & & \frac{\sqrt{15}}{4} & \frac{1}{4} & & & & & & & & & & & \\
& & & & & \frac{1}{2} & \frac{\sqrt{3}}{2} & & & & & & & & & \\
& & & & & \frac{\sqrt{3}}{2} & -\frac{1}{2} & & & & & & & & & \\
& & & & & & & -\frac{1}{2} & & & & & & & & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & & & & & & & & -\frac{1}{2} & & & & & & & \\
& & & & & & & & & \frac{1}{2} & \frac{\sqrt{3}}{2} & & & & & \\
& & & & & & & & & \frac{\sqrt{3}}{2} & -\frac{1}{2} & & & & & \\
& & & & & & & & & & & -\frac{1}{4} & \frac{\sqrt{15}}{4} & & & \\
& & & & & & & & & & & \frac{\sqrt{15}}{4} & \frac{1}{4} & & & \\
& & & & & & & & & & & & & -1 & & \\
& & & & & & & & & & & & & & 1 & \\
& & & & & & & \frac{\sqrt{3}}{2} & & & & & & & & -\frac{1}{2}
\end{vmatrix}
$$

$$
(34) = \left|\begin{array}{cccccccccccccccc}
-\frac{1}{3} & \frac{\sqrt{8}}{3} &&&&&&&&&&&&&& \\
\frac{\sqrt{8}}{3} & \frac{1}{3} &&&&&&&&&&&&&& \\
&& \frac{1}{3} && \frac{\sqrt{8}}{3} &&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&& \frac{\sqrt{8}}{3} && -\frac{1}{3} &&&&&&&&&&& \\
&&&&& -1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& 1 &&&&&&&& \\
&&&&&&&& -1 &&&&&&& \\
&&&&&&&&& -1 &&&&&& \\
&&&&&&&&&& 1 &&&&& \\
&&&&&&&&&&& \frac{1}{3} && \frac{\sqrt{8}}{3} && \\
&&&&&&&&&&&& -1 &&& \\
&&&&&&&&&&& \frac{\sqrt{8}}{3} && -\frac{1}{3} && \\
&&&&&&&&&&&&&& -\frac{1}{3} & \frac{\sqrt{8}}{3} \\
&&&&&&&&&&&&&& \frac{\sqrt{8}}{3} & \frac{1}{3}
\end{array}\right|
$$

$$
(45) = \left|\begin{array}{cccccccccccccccc}
\frac{1}{2} &&&&& \frac{\sqrt{3}}{2} &&&&&&&&&& \\
& -1 &&&&&&&&&&&&&& \\
&& -1 &&&&&&&&&&&&& \\
&&& \frac{1}{2} &&&&&&&& \frac{\sqrt{3}}{2} &&&& \\
&&&& \frac{1}{2} &&&&&&&& \frac{\sqrt{3}}{2} &&& \\
\frac{\sqrt{3}}{2} &&&&& -\frac{1}{2} &&&&&&&&&& \\
&&&&&& -\frac{1}{2} && \frac{\sqrt{3}}{2} &&&&&&& \\
&&&&&&& -\frac{1}{2} && \frac{\sqrt{3}}{2} &&&&&& \\
&&&&&& \frac{\sqrt{3}}{2} && -\frac{1}{2} &&&&&&& \\
&&&&&&& \frac{\sqrt{3}}{2} && \frac{1}{2} &&&&&& \\
&&&&&&&&&& \frac{1}{2} &&&&& \frac{\sqrt{3}}{2} \\
&&& \frac{\sqrt{3}}{2} &&&&&&&& -\frac{1}{2} &&&& \\
&&&& \frac{\sqrt{3}}{2} &&&&&&&& -\frac{1}{2} &&& \\
&&&&&&&&&&&&& 1 && \\
&&&&&&&&&&&&&& 1 & \\
&&&&&&&&&& \frac{\sqrt{3}}{2} &&&&& \frac{1}{2}
\end{array}\right|
$$

$$
(56) = \left|\begin{array}{cccccccccccccccc}
-1 &&&&&&&&&&&&&&& \\
& -1 &&&&&&&&&&&&&& \\
&& -1 &&&&&&&&&&&&& \\
&&& -1 &&&&&&&&&&&& \\
&&&& -1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& 1 &&&&&&&& \\
&&&&&&&& -1 &&&&&&& \\
&&&&&&&&& -1 &&&&&& \\
&&&&&&&&&& -1 &&&&& \\
&&&&&&&&&&& 1 &&&& \\
&&&&&&&&&&&& 1 &&& \\
&&&&&&&&&&&&& 1 && \\
&&&&&&&&&&&&&& 1 & \\
&&&&&&&&&&&&&&& 1
\end{array}\right|
$$

References

[1] Yamanouchi T 1937 *Proc. Phys. Math. Soc. Japan* **18** 623

[2] Rutherford D E 1948 *Substitutional Analysis* (Edinburgh: Edinburgh University Press).

[3] Hamermesh M 1962 *Group Theory and its Application to Physical Problems* (Reading, MA: Addison-Wesley)

[4] Kaplan I 1975 *Symmetry of Many-electron Systems* (New York: Academic)

[5] Horie H 1964 *J. Phys. Soc. Japan* **19** 1783

[6] Suryanarayana Ch and Kondala Rao M 1982 *J. Phys. A: Math. Gen.* **15** 2013

[7] Wu W and Zhang Q 1992 *J. Phys. A: Math. Gen.* **25** 3737

[8] Wu W and Zhang Q 1994 *Int. J. Quant. Chem.* **50** 55

[9] Schützenberger M P 1963 *Math. Scand.* **12** 117

[10] Schützenberger M P 1977 *Combinatoire et Représentation du Groupe Symétrique (Lecture Notes in Mathematics 579)* ed D Foata (New York: Springer) p 59

[11] Thomas G 1977 *Discrete Math.* **17** 107–18

[12] Knuth D E 1973 *The Art of Computer Programming* vol 3 (Reading, MA: Addison-Wesley)

[13] Young A 1977 *The Collected Papers of Alfred Young* (Toronto: University of Toronto Press)